

The BV Formalization of Chern-Simons Theory on Deformed Superspace

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Abstract

In this paper we will study non-abelian Chern-Simons theory on a deformed superspace. We will deform the superspace in such a way that it includes the noncommutativity between bosonic and fermionic coordinates. We will first analyse the BRST and the anti-BRST symmetries of the Chern-Simons theory on this deformed superspace. Then we will analyse the extended BRST and the extended anti-BRST symmetries of this theory in the Batalin-Vilkovisky (BV) formalism. Finally, we will express these extended BRST and extended anti-BRST symmetries in extended superspace formalism by introducing new Grassmann coordinates.

Key Words: Noncommutative superspace, Batalin-Vilkovisky formalism.

1 Introduction

Highly supersymmetric Chern-Simons theories are important because they are thought to describe the world-volume of $M2$ -membranes in M -theory, at low energies [1]-[5]. In fact, the world-volume of $M2$ -membranes in M -theory, at low energies, is thought to be described by a superconformal Chern-Simons-matter theory with manifest $\mathcal{N} = 8$ supersymmetry [6]. A Lie

3-algebra has been used to construct an action for this theory [2, 3]. However, only one example of such a 3-algebra exists and so far the rank of the gauge group has not been increased.

A $U(N) \times U(N)$ superconformal Chern-Simons-matter theory with level k and $-k$ is also thought to describe the world-volume of N $M2$ -membranes placed at the singularity of R^8/Z_k [8]. This theory allows arbitrary rank and but only has $\mathcal{N} = 6$ supersymmetry. However, by utilizing monopole operators this symmetry gets enhanced to $\mathcal{N} = 8$ supersymmetry. Furthermore, after the enhancement of the supersymmetry this theory possess a $SO(8)$ R -symmetry at Chern-Simons levels $k = 1, 2$ [7].

Chern-Simons theory in $\mathcal{N} = 1$ superfield formalism has also been used in analysing the action of $M2$ -membranes [9]. This Chern-Simons theory in $\mathcal{N} = 1$ superfield formalism reproduces the full Bagger and Lambert theory [2, 3] for a 3-algebra with totally antisymmetric structure constants. In doing so octonionic self-dual tensors are used in the construction of the real super-potential. The super-potential obtained in this way is only manifestly $SO(7)$ invariant. However, for specially chosen couplings, the component action coincides with the Bagger and Lambert action [2, 3], and hence full $SO(8)$ symmetry is restored. Thus, although the octonionic self-dual tensors are only $SO(7)$ invariant, they can be made $SO(8)$ invariant by a special choice of the parameters in this particular action.

Chern-Simons theory with $\mathcal{N} = 1$ supersymmetry has also been studied in relation to axion gauge symmetry [10]. This occurs in the supergravity theories arising from flux compactifications of superstrings. This also occurs in the Scherk-Schwarz generalized dimensional reduction in M -theory. In these theories the mass term arises through a Higgs mechanism and the supergravity description corresponds to the gauging of some axion symmetries. These are related to shifts of the scalar fields coming from wrapped RR forms or from the NS two-form B field in type II strings, and from the wrapped three-form in M -Theory. Chern-Simons theory with $\mathcal{N} = 1$ supersymmetry coupled to parity-preserving matter fields has also been analysed using the Parkes-Siegel formulation [11].

It is expected that string theory may introduce noncommutativity in spacetime and so field theories with spacetime noncommutativity have been thoroughly studied [12]-[17]. Noncommutative tori have also been used to study compactification in M -theory [18]. Noncommutativity also arises by viewing M -theory as the $N \rightarrow \infty$ limit of the supersymmetric matrix quantum mechanics describing $D0$ -branes [19]. Furthermore, the NS antisymmet-

ric tensor background is a source of spacetime noncommutativity in string theory [20, 21].

The extension of spacetime noncommutativity to superspace noncommutativity is related to the presence of other background fields. The RR field strength background give rise to θ - θ type deformations [22, 23]. Furthermore, x - θ deformation is caused by a gravitino background [24]. Thus, superspace noncommutativity also arises in string theory. So, field theories with superspace noncommutativity have also been thoroughly studied [23]-[29].

The BRST symmetry for the Chern-Simons theory has also been investigated [30, 31]. The BRST symmetry of $\mathcal{N} = 1$ abelian Chern-Simons theory [32] and $\mathcal{N} = 1$ non-abelian Chern-Simons theory [33] has been discussed in the superspace formalism. These theories can also be analysed in the background field method. In this method, all the fields in a theory are shifted. The BRST and the anti-BRST symmetries of these shifted fields can then be analysed in the Batalin-Vilkovisky formalism [34]-[40]. In this formalism the extended BRST and the extended anti-BRST symmetries arise due to the invariance of a theory under both the original BRST and the original anti-BRST transformations along with these shift transformation. This has been done for the conventional Yang-Mills theories and the conventional Chern-Simons theories [41]-[47]. Furthermore, the extended BRST and the extended anti-BRST symmetries have been analysed in the extended superspace formalism [48, 49]. This is because the BRST and the anti-BRST symmetries mix the fermionic and bosonic coordinates and can thus be viewed as supersymmetric transformations.

In this paper we will analyse the $\mathcal{N} = 1$ Chern-Simons theory in a deformed superspace. The deformation of the superspace will break the supersymmetry of the theory. Then we will analyse the extended BRST and the extended anti-BRST symmetries for this Chern-Simons theory in the BV-formalism. Finally we shall express our results in an extended superspace formulation by introducing new Grassmann coordinates. It may be noted even though the results of this paper can easily be first generalized and then used to analyse the deformation of the ABJM theory, this will not be done here. In fact, in this paper, we will only analyse the pure Chern-Simons theory with no matter fields.

2 Deformed Super-Chern-Simons Theory

In this section we shall construct a three dimensional Chern-Simons theory on a deformed superspace. To do so we define θ^a as a two-component Grassmann parameter and let $y^\mu = x^\mu + \theta^a (\gamma^\mu)_a^b \theta_b$. Then we promote them to operators $\hat{\theta}^a$ and \hat{y}^μ and impose the following deformation of the superspace algebra,

$$\begin{aligned} \{\hat{\theta}^a, \hat{\theta}^b\} &= C^{ab}, & [\hat{y}^\mu, \hat{y}^\nu] &= B^{\mu\nu}, \\ [\hat{y}^\mu, \hat{\theta}^a] &= A^{\mu a}. \end{aligned} \quad (1)$$

We use Weyl ordering and express the Fourier transformation of this superfield as,

$$\hat{\Gamma}_a(\hat{y}, \hat{\theta}) = \int d^3k \int d^2\pi e^{-ik\hat{y} - \pi\hat{\theta}} \Gamma_a(k, \pi). \quad (2)$$

Now we have a one to one map between a function of $\hat{\theta}, \hat{y}$ to a function of ordinary superspace coordinates θ, y via

$$\Gamma_a(y, \theta) = \int d^3k \int d^2\pi e^{-iky - \pi\theta} \Gamma_a(k, \pi), \quad (3)$$

where

$$\Gamma_a(y, \theta) = \chi_a + B\theta_a + \frac{1}{2}(\gamma^\mu \theta A_\mu)_a + i\theta^2 \left[\lambda_a - \left(\frac{1}{2} \gamma^\mu \partial_\mu \chi \right)_a \right]. \quad (4)$$

We can express the product of two fields $\hat{\Gamma}^a(\hat{y}, \hat{\theta}) \hat{\Gamma}_a(\hat{y}, \hat{\theta})$ on this deformed superspace as

$$\begin{aligned} \hat{\Gamma}^a(\hat{y}, \hat{\theta}) \hat{\Gamma}_a(\hat{y}, \hat{\theta}) &= \int d^3k_1 d^3k_2 \int d^2\pi_1 d^2\pi_2 \exp -i((k_1 + k_2)\hat{y} + (\pi_1 + \pi_2)\hat{\theta}) \\ &\quad \times \exp(i\Delta) \Gamma^a(k_1, \pi_1) \Gamma_a(k_2, \pi_2), \end{aligned} \quad (5)$$

where

$$\exp(i\Delta) = \exp -\frac{i}{2} (C^{ab} \pi_a^1 \pi_b^2 + B^{\mu\nu} k_\mu^2 k_\nu^1 + A^{\mu a} (\pi_a^2 k_\mu^1 - k_\mu^2 \pi_a^1)), \quad (6)$$

So we can now define the star product between ordinary functions as follows:

$$\begin{aligned} \Gamma^a(y, \theta) \star \Gamma_a(y, \theta) &= \exp -\frac{i}{2} (C^{ab} \partial_a^1 \partial_b^2 + B^{\mu\nu} \partial_\mu^2 \partial_\nu^1 + A^{\mu a} (\partial_a^2 \partial_\mu^1 - \partial_\mu^2 \partial_a^1)) \\ &\quad \times \Gamma^a(y_1, \theta_1) \Gamma_a(y_2, \theta_2) \big|_{y_1=y_2=y, \theta_1=\theta_2=\theta}. \end{aligned} \quad (7)$$

The star product reduces to the usual Moyal star product for the bosonic noncommutativity in the limit $C^{ab} = A^{a\mu} = 0$ and for $A^{a\mu} = C^{ab} = 0$ it reduces to the standard fermionic star product. It is also useful to define the following bracket

$$[\Gamma^a \star \Gamma_a] = \frac{1}{2} T_A f_{BC}^A \Gamma^{aB} \star \Gamma_a^C. \quad (8)$$

In order to construct a Chern-Simons theory on this deformed superspace, it is useful to define the following fields

$$\Omega_a = \omega_a - \frac{1}{6} [\Gamma^b \star \Gamma_{ab}], \quad (9)$$

$$\omega_a = \frac{1}{2} D^b D_a \Gamma_b - \frac{i}{2} [\Gamma^b \star D_b \Gamma_a] - \frac{1}{6} [\Gamma^b \star [\Gamma_b \star \Gamma_a]], \quad (10)$$

$$\Gamma_{ab} = -\frac{i}{2} (D_{(a} \Gamma_{b)} - i [\Gamma_a \star \Gamma_b]), \quad (11)$$

where

$$D_a = \partial_a + (\gamma^\mu \theta \partial_\mu)_a. \quad (12)$$

Now the non-abelian Chern-Simons theory on this deformed superspace can now be written as

$$\mathcal{L}_c = \int d^2 \theta \, Tr(\Gamma^a \star \Omega_a)|. \quad (13)$$

and $'|'$ means that the quantity is evaluated at $\theta_a = 0$. We consider our theory to be defined on a manifold without a boundary. This theory on an undeformed superspace has $\mathcal{N} = 1$ supersymmetry, however on the deformed superspace the supersymmetry is broken. This is because of the deformation $\{\hat{\theta}^a, \hat{\theta}^b\} = C^{ab}$, the supersymmetry corresponding to Q_a is broken [23]-[29]. Here Q_a is the generator of $\mathcal{N} = 1$ supersymmetry and is given by

$$Q_a = \partial_a - (\gamma^\mu \theta \partial_\mu)_a. \quad (14)$$

For theories with $\mathcal{N} = 2$ supersymmetry in three dimensions we have two super-charges. It is thus possible to deform the super-algebra corresponding to one of them and leave super-algebra corresponding to the other one undeformed. So we, can break the supersymmetry from $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry by such deformations. In four dimensions, we can also break the supersymmetry to $\mathcal{N} = 1/2$ supersymmetry. However, for $\mathcal{N} = 1$ supersymmetry in three dimensions such a deformation breaks all the supersymmetry of the theory.

All the degrees of freedom in this Lagrangian density are not physical as it is invariant under the following super-gauge transformations

$$\delta\Gamma_a = (D_a - i\Gamma_a) \star \Lambda, \quad (15)$$

where $\Lambda = \Lambda^A T_A$. In order to quantise this theory we will have to add a gauge fixing term and a ghost term to it. We choose the gauge fixing term \mathcal{L}_{gf} and the ghost term \mathcal{L}_{gh} as follows:

$$\begin{aligned} \mathcal{L}_{gf} &= \int d^2\theta \, Tr(F \star D^a \Gamma_a)|, \\ \mathcal{L}_{gh} &= \int d^2\theta \, Tr(\bar{c} D^a (D_a - i\Gamma_a) \star c)|. \end{aligned} \quad (16)$$

Here B is a matrix valued scalar superfield, c and \bar{c} are matrix valued anti-commuting superfields. These superfields are suitably contracted with generators of the Lie algebra in the adjoint representation,

$$\begin{aligned} F(y, \theta) &= F^A(y, \theta) T_A, \quad c(y, \theta) = c^A(y, \theta) T_A, \\ \bar{c}(y, \theta) &= \bar{c}^A(y, \theta) T_A. \end{aligned} \quad (17)$$

The total Lagrangian density which is obtained by the sum of the original Lagrangian density with the gauge fixing term and the ghost term is invariant under the following BRST transformations

$$\begin{aligned} s\Gamma_a &= (D_a - i\Gamma_a) \star c, \quad sc = -\frac{1}{2}[c \star c], \\ s\bar{c} &= -F, \quad sF = 0, \end{aligned} \quad (18)$$

where

$$[c \star c] = \frac{1}{2} T_A f_{BC}^A c^B \star c^C. \quad (19)$$

In fact, this total Lagrangian density is also invariant under the following anti-BRST transformations

$$\begin{aligned} \bar{s}\Gamma_a &= (D_a - i\Gamma_a) \star \bar{c}, \quad \bar{s}c = F - [c \star \bar{c}], \\ \bar{s}\bar{c} &= -\frac{1}{2}[\bar{c} \star \bar{c}], \quad \bar{s}F = [F \star \bar{c}]. \end{aligned} \quad (20)$$

These transformations satisfy

$$s^2 = \bar{s}^2 = 0. \quad (21)$$

In fact, they also satisfy $s\bar{s} + \bar{s}s = 0$. Now the sum of the gauge fixing term and the ghost term can be written as

$$\begin{aligned}\mathcal{L}_{gf} + \mathcal{L}_{gh} &= - \int d^2\theta \frac{s\bar{s}}{2} Tr(\Gamma^a \star \Gamma_a)| \\ &= \int d^2\theta \frac{\bar{s}s}{2} Tr(\Gamma^a \star \Gamma_a)|.\end{aligned}\tag{22}$$

The BRST and the anti-BRST invariance of the total Lagrangian density given by

$$\mathcal{L} = \mathcal{L}_c + \mathcal{L}_{gf} + \mathcal{L}_{gh},\tag{23}$$

follows from the fact that these BRST and the anti-BRST transformations are nilpotent and the sum of the gauge fixing term and the ghost term can be written as a total BRST and a total anti-BRST variation. For the original classical Lagrangian density the BRST and anti-BRST transformations are just gauge transformations with Λ replaced by the ghosts or the anti-ghosts. Thus, the total Lagrangian density is invariant under the BRST and the anti-BRST transformations.

3 Extended BRST

In this section we will analyse the extended BRST invariant Lagrangian density. To do so we first shift the original fields as

$$\begin{aligned}\Gamma_a &\rightarrow \Gamma_a - \tilde{\Gamma}_a, \\ c &\rightarrow c - \tilde{c}, \\ \bar{c} &\rightarrow \bar{c} - \tilde{\bar{c}}, \\ F &\rightarrow F - \tilde{F}.\end{aligned}\tag{24}$$

The extended BRST invariant Lagrangian density is obtained by requiring it to be invariant under both the original BRST transformations and these shift transformations of the original fields

$$\tilde{\mathcal{L}} = \mathcal{L}(\Gamma_a - \tilde{\Gamma}_a, c - \tilde{c}, \bar{c} - \tilde{\bar{c}}, F - \tilde{F}).\tag{25}$$

In order to discuss the extended BRST invariant Lagrangian density it will be useful to define $\tilde{\nabla}_a$ as

$$\tilde{\nabla}_a = D_a - i\Gamma_a + i\tilde{\Gamma}_a.\tag{26}$$

Now the extended BRST invariant Lagrangian density is invariant under the following extended BRST transformations

$$s\Gamma_a = \psi_a, \quad s\tilde{\Gamma}_a = \psi_a - \tilde{\nabla}_a \star (c - \tilde{c}), \quad (27)$$

$$\begin{aligned} sc = \epsilon, \quad s\tilde{c} &= \epsilon + \frac{1}{2}[(c - \tilde{c}) \star (c - \tilde{c})], \\ s\bar{c} = \bar{\epsilon}, \quad s\tilde{\bar{c}} &= \bar{\epsilon} + (F - \bar{F}), \\ sF = \psi, \quad s\tilde{F} &= \psi, \end{aligned} \quad (28)$$

where

$$\tilde{\nabla}_a \star (c - \tilde{c}) = D_a \star (c - \tilde{c}) - i(\Gamma_a - \tilde{\Gamma}_a) \star (c - \tilde{c}). \quad (29)$$

Here $\psi_a, \epsilon, \bar{\epsilon}$ and ψ are the ghost fields associated with the shift symmetries of the original fields Γ_a, c, \bar{c} and F respectively. The BRST transformations of these ghosts associated with the shift symmetry vanish,

$$\begin{aligned} s\psi_a &= 0, \quad s\epsilon = 0, \\ s\bar{\epsilon} &= 0, \quad s\psi = 0, \end{aligned} \quad (30)$$

We add anti-fields with opposite parity to the original fields. These anti-fields transform into new auxiliary fields b_a, B, \bar{B} and b under BRST transformations,

$$\begin{aligned} s\Gamma_a^* &= -b_a, \quad sc^* = -B, \\ s\bar{c}^* &= -\bar{B}, \quad sF^* = -b. \end{aligned} \quad (31)$$

The BRST transformations of these new auxiliary fields also vanish

$$\begin{aligned} sb_a &= 0, \quad sB = 0, \\ s\bar{B} &= 0, \quad sb = 0. \end{aligned} \quad (32)$$

We now choose the Lagrangian density to gauge fix the shift symmetry in such a way that the tilde fields will be made to vanish so that we can recover the original theory

$$\begin{aligned} \tilde{\mathcal{L}}_{gf} + \tilde{\mathcal{L}}_{gh} &= \int d^2\theta \operatorname{Tr}(-b^a \star \tilde{\Gamma}_a + \Gamma^{*a} \star (\psi_a - \tilde{\nabla}_a \star (c - \tilde{c})) \\ &\quad - \bar{B} \star \tilde{c} + \bar{c}^* \star \left(\epsilon + \frac{1}{2}[(c - \tilde{c}) \star (c - \tilde{c})] \right) \\ &\quad + B \star \tilde{\bar{c}} - c^* \star (\bar{\epsilon} + (F - \tilde{F})) + b \star \tilde{F} + F^* \star \psi)_|. \end{aligned} \quad (33)$$

Here tilde fields vanish upon integrating out the auxiliary fields b_a, B, \bar{B} and b . This Lagrangian density is invariant under the original BRST transformation and the shift transformations. Along with this Lagrangian density we have the original Lagrangian density, which is only a function of the original fields. So we define Ψ as

$$\mathcal{L}'_{gf} + \mathcal{L}'_{gh} = \int d^2\theta \, Tr(s\Psi)|. \quad (34)$$

Expanding this Lagrangian density, we obtain

$$\mathcal{L}'_{gf} + \mathcal{L}'_{gh} = \int d^2\theta \, Tr \left(-\frac{\delta\Psi}{\delta\Gamma_a} \star \psi_a + \frac{\delta\Psi}{\delta c} \star \epsilon + \frac{\delta\Psi}{\delta\bar{c}} \star \bar{\epsilon} - \frac{\delta\Psi}{\delta F} \star \psi \right)|. \quad (35)$$

If we integrate out the fields setting the tilde fields to zero, we have

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_c + \tilde{\mathcal{L}}_{gf} + \tilde{\mathcal{L}}_{gh} + \mathcal{L}'_{gf} + \mathcal{L}'_{gh} \\ &= \mathcal{L}_c(\Gamma_a - \tilde{\Gamma}_a) + \int d^2\theta \, Tr \left(\Gamma^{*a} \star \nabla_a c + \frac{1}{2} \bar{c}^* \star [c \star c] - c^* \star F \right. \\ &\quad \left. - \left(\Gamma^{*a} + \frac{\delta\Psi}{\delta\Gamma_a} \right) \star \psi^a + \left(\bar{c}^* + \frac{\delta\Psi}{\delta c} \right) \star \epsilon - \left(c^* + \frac{\delta\Psi}{\delta\bar{c}} \right) \star \bar{\epsilon} \right. \\ &\quad \left. + \left(F^* - \frac{\delta\Psi}{\delta F} \right) \star \psi \right)|. \end{aligned} \quad (36)$$

The explicit expression for the anti-fields is achieved by integrating out the ghosts associated with the shift symmetry,

$$\begin{aligned} \Gamma^{*a} &= -\frac{\delta\Psi}{\delta\Gamma_a}, & \bar{c}^* &= -\frac{\delta\Psi}{\delta c}, \\ c^* &= \frac{\delta\Psi}{\delta\bar{c}}, & F^* &= \frac{\delta\Psi}{\delta F}. \end{aligned} \quad (37)$$

With these identifications we obtain an explicit form for the Lagrangian density which is invariant under the extended BRST transformations.

3.1 Extended Anti-BRST Lagrangian

In the previous sections we analysed the extended BRST symmetry. Now we will discuss the extended anti-BRST symmetry. The original Lagrangian

density is also invariant under the following extended anti-BRST transformations,

$$\begin{aligned}
\bar{s}\Gamma_a &= \Gamma_a^* + \tilde{\nabla}_a \star (c - \tilde{c}), \\
\bar{s}c &= c^* + (F - \bar{F}) - [(c - \tilde{c}) \star (\bar{c} - \tilde{\bar{c}})], \\
\bar{s}\bar{c} &= \bar{c}^* - \frac{1}{2}[(\bar{c} - \tilde{\bar{c}}) \star (\bar{c} - \tilde{\bar{c}})], \\
\bar{s}F &= F^* + [(F - \tilde{F}) \star (\bar{c} - \tilde{\bar{c}})],
\end{aligned} \tag{38}$$

and shifted super-fields have the following extended anti-BRST transformations,

$$\begin{aligned}
\bar{s}\tilde{\Gamma}_a &= \Gamma_a^*, & \bar{s}\tilde{c} &= c^*, \\
\bar{s}\tilde{\bar{c}} &= \bar{c}^*, & \bar{s}\tilde{F} &= F^*.
\end{aligned} \tag{39}$$

The ghost fields associated with the shift symmetry have the following extended anti-BRST transformations,

$$\begin{aligned}
\bar{s}\psi_a &= b_a + \tilde{\nabla}_a \star (F - \tilde{F}) - [(\tilde{\nabla}_a \star (c - \tilde{c})) \star (\bar{c} - \tilde{\bar{c}})], \\
\bar{s}\epsilon &= B - [(F - \tilde{F}) \star (\bar{c} - \tilde{\bar{c}})] + [[(\bar{c} - \tilde{\bar{c}}) \star (c - \tilde{c})] \star (c - \tilde{c})], \\
\bar{s}\bar{\epsilon} &= \bar{B} - [(F - \tilde{F}) \star (\bar{c} - \tilde{\bar{c}})], \\
\bar{s}\psi &= b.
\end{aligned} \tag{40}$$

The extended anti-BRST transformations of the anti-fields and the auxiliary fields associated with the shift symmetry vanish,

$$\begin{aligned}
\bar{s}b_a &= 0, & \bar{s}\Gamma_a^* &= 0, \\
\bar{s}B &= 0, & \bar{s}c^* &= 0, \\
\bar{s}\bar{B} &= 0, & \bar{s}\bar{c}^* &= 0, \\
\bar{s}\bar{b} &= 0, & \bar{s}F^* &= 0.
\end{aligned} \tag{41}$$

For the Lagrangian density, which is both BRST and anti-BRST invariant, it follows that it must also be invariant under the extended anti-BRST transformation at least on-shell, where these transformations reduce to the original anti-BRST transformations.

3.2 Extended Superspace Formulation

The BRST transformations and the anti-BRST transformations are also supersymmetry transformations as they mix bosonic and fermionic fields. In

fact the BRST and the anti-BRST transformations have been expressed in superspace even for bosonic theories [48, 49]. We will now express the extended BRST and the extended anti-BRST invariance of this Chern-Simons theory in superspace. This superspace has nothing to do with the original superspace and a similar structure will exist even for a bosonic theory. It is related to the supersymmetric nature of the extended BRST and the extended anti-BRST symmetries. Thus we introduce two anti-commuting Grassmann parameters ξ and $\bar{\xi}$ and define the following superfields with them,

$$\begin{aligned}
\phi_a(x, \xi, \bar{\xi}) &= \Gamma_a + \xi\psi_a + \bar{\xi}(\Gamma_a^* + \tilde{\nabla}_a \star (\bar{c} - \tilde{c})) \\
&\quad + \xi\bar{\xi}(b_a + \tilde{\nabla}_a \star (F - \tilde{F}) - [(\tilde{\nabla}_a \star (c - \tilde{c})) \star (\bar{c} - \tilde{c})]), \\
\tilde{\phi}_a(x, \xi, \bar{\xi}) &= \tilde{\Gamma}_a + \xi(\psi_a - \tilde{\nabla}_a \star (c - \tilde{c}) + \bar{\xi}\Gamma_a^* + \xi\bar{\xi}b_a, \\
\eta(x, \xi, \bar{\xi}) &= c + \xi\epsilon + \bar{\xi}(c^* + (F - \tilde{F}) - [(c - \tilde{c}) \star (\bar{c} - \tilde{c})]) \\
&\quad + \xi\bar{\xi}(B - [(F - \tilde{F}) \star (c - \tilde{c})] + [(\bar{c} - \tilde{c}) \star (c - \bar{c})] \star (c - \bar{c})), \\
\tilde{\eta}(x, \xi, \bar{\xi}) &= \tilde{c} + \xi(\epsilon + \frac{1}{2}[(c - \tilde{c}) \star (c - \tilde{c})]) + \bar{\xi}c^* + \xi\bar{\xi}B, \\
\bar{\eta}(x, \xi, \bar{\xi}) &= \bar{c} + \xi\bar{c} + \bar{\xi}(\bar{c}^* - \frac{1}{2}[(\bar{c} - \tilde{c}) \star (\bar{c} - \tilde{c})]) \\
&\quad + \xi\bar{\xi}(\bar{B} - [(F - \tilde{F}) \star (\bar{c} - \tilde{c})]), \\
\tilde{\bar{\eta}}(x, \xi, \bar{\xi}) &= \tilde{\bar{c}} + \xi(\bar{c} + (F - \tilde{F})) + \bar{\xi}\bar{c}^* + \xi\bar{\xi}\bar{B}.
\end{aligned} \tag{42}$$

Now we have

$$\begin{aligned}
-\frac{1}{2} \int d\xi d\bar{\xi} d^2\theta \text{Tr}(\tilde{\phi}^a \star \tilde{\phi}_a)_| &= \int d^2\theta \text{Tr}(-\Gamma^{*a} \star (\psi_a - \tilde{\nabla}_a \star (c - \tilde{c})) \\
&\quad - b_a \star \tilde{\Gamma}^a)_|, \\
\int d\xi d\bar{\xi} d^2\theta \text{Tr}(\tilde{\eta} \star \tilde{\eta})_| &= \int d^2\theta \text{Tr} \left(-\bar{B}\tilde{c} + B \star \tilde{c} - \bar{B} \star \tilde{c} \right. \\
&\quad \left. + \bar{c}^* \star \left(\epsilon + \frac{1}{2}[(c - \tilde{c}) \star (c - \tilde{c})] \right) \right. \\
&\quad \left. - c^* \star (\bar{c} + (F - \tilde{F})) \right)_|.
\end{aligned} \tag{43}$$

Thus the sum of the shifted gauge fixing term and the shifted ghosts term can be written as,

$$\begin{aligned}
\tilde{\mathcal{L}}_{gf} + \tilde{\mathcal{L}}_{gf} &= \int d\xi d\bar{\xi} d^2\theta \text{Tr} \left(-\frac{1}{2} \tilde{\phi}^a \star \tilde{\phi}_a + \tilde{\eta} \star \tilde{\bar{\eta}} \right)_| \\
&= \int d^2\theta \text{Tr} \left(-b^a \star \tilde{\Gamma}_a - \Gamma^{*a} \star (\psi_a - \nabla_a \star (c - \tilde{c})) \right. \\
&\quad \left. - \bar{B} \star \tilde{c} + \bar{c}^* \star \left(\epsilon + \frac{1}{2} [(c - \tilde{c}) \star (c - \tilde{c})] \right) \right. \\
&\quad \left. + B \star \tilde{\bar{c}} - c^* \star (\bar{\epsilon} + (F - \tilde{F})) \right)_|.
\end{aligned} \tag{44}$$

Being the $\xi\bar{\xi}$ component of a superfield, this gauge fixing Lagrangian is manifestly invariant under extended BRST and anti-BRST transformation. We now note that we can choose a fermionic superfield such that [48],

$$\Phi(x, \xi, \bar{\xi}) = \Psi + \xi s\Psi + \bar{\xi} \bar{s}\Psi + \xi\bar{\xi} s\bar{s}\Psi. \tag{45}$$

The $\xi\bar{\xi}$ component of this equation vanishes after using the equations of motion. This is expected as the sum of the original gauge fixing term and the ghost term is invariant under the BRST and the anti-BRST transformations. Thus the $\xi\bar{\xi}$ component of $\Phi(x, \xi, \bar{\xi})$ vanish on shell because when we use the equations of motion, the tilde fields vanish and the theory reduces to the original theory which is invariant under the BRST and the anti-BRST transformations. We can now express the sum of the original gauge fixing term and the ghost term as,

$$\mathcal{L}'_{gf} + \mathcal{L}'_{gf} = \int d\xi d\bar{\xi} d^2\theta \text{Tr} (\delta(\bar{\xi}) \Phi(x, \xi, \bar{\xi}))_|. \tag{46}$$

This is not only invariant under extended BRST transformations but it is also invariant under extended anti-BRST transformations on-shell. The total Lagrangian density is invariant under extended BRST transformations. It is also invariant under extended anti-BRST transformations on-shell. We now write this total Lagrangian density as,

$$\begin{aligned}
\tilde{\mathcal{L}} &= \mathcal{L}_c + \tilde{\mathcal{L}}_{gf} + \tilde{\mathcal{L}}_{gf} + \mathcal{L}'_{gf} + \mathcal{L}'_{gf} \\
&= \mathcal{L}_c(\Gamma_a - \tilde{\Gamma}_a) + \int d\xi d\bar{\xi} d^2\theta \text{Tr} \left(-\frac{1}{2} \tilde{\phi}^a \star \tilde{\phi}_a + \tilde{\eta} \star \tilde{\bar{\eta}} \right)_| \\
&\quad + \int d\xi d\bar{\xi} d^2\theta \text{Tr} (\delta(\bar{\xi}) \Phi(x, \xi, \bar{\xi}))_|.
\end{aligned} \tag{47}$$

As F and \tilde{F} are auxiliary fields so we redefine them as $F - \tilde{F} \rightarrow F$ and integrated out $(F + \tilde{F})$. The constant thus obtained can be absorbed into the normalisation constant.

3.3 Conclusion

In this paper we analysed the Chern-Simons theory in deformed superspace where the deformation included the noncommutativity between bosonic coordinates and fermionic coordinates. We found that the sum of the original classical Lagrangian density, a gauge fixing term and a ghost term was invariant under the BRST and the anti-BRST transformations. We also analysed the extended BRST and the extended anti-BRST symmetries of this theory in the extended superspace formalism. This theory was found to be invariant under extended BRST transformations. It was also found to be invariant under on-shell extended anti-BRST transformations.

The spacelike noncommutative field theories are known to be unitarity [50]-[51]. However, due to Eq. (7), infinite temporal derivatives will occur in the product of fields for field theories with spacetime noncommutativity. It is well known that the evolution of the S -matrix is not unitary for the field theories with higher order temporal derivatives [52]-[55]. Thus, spacetime noncommutativity will break the unitarity of the resultant theory. However, if we restrict the theory to spacelike noncommutativity and thus do not include any higher order temporal derivatives then this problem can be avoided. In fact, in the case of spacelike noncommutativity it is possible to construct the Northey's charges [56]-[57]. Thus, if we restrict the spacetime deformations to spacelike noncommutativity then we can construct the Northey's charges for this deformed theory. It will be interesting to construct the BRST and the anti-BRST charges for this theory and use them to find the physical states in this theory.

It will be interesting to generalise the result of this paper to Chern-Simons theories with higher supersymmetry, coupled to matter fields. In particular, the analysis of $U(N) \times U(N)$ Chern-Simons gauge theory, with level k and $-k$ enhanced to $\mathcal{N} = 8$ supersymmetry for $k = 1, 2$, suitably coupled to matter fields in this deformed superspace will have important consequences for M -theory. Due to the duality between M -theory and II string theory, we expect that a noncommutative deformation of the super algebra on the string theory side will correspond to some deformation of the super algebra on the M -theory side. In fact, just like a background two-form field strength be-

comes a source of noncommutativity for D -branes, a background three-form field suitably coupled to the ABJM theory could also lead to the noncommutativity. It may be noted that a three-form field strength occurs naturally in $M5$ -branes. Furthermore, $M5$ -branes in M -theory act as analogous objects to a D -brane in string theory, in the sense that $M2$ -branes can end on them. Thus, the coupling of ABJM theory to a background three-form field strength can be useful in describing the physics of $M2$ -branes ending on $M5$ -branes. As the action for a single $M5$ -brane can be derived by demanding the κ -symmetry of the open membrane ending on it [58], the analysis of ABJM theory coupled to a background three-form field strength might give some useful insights into understanding the dynamics of multiple $M5$ -branes. It may be noted that even though the action for a single $M5$ -brane is known, the action for multiple $M5$ -branes is not known [59]-[63]. Coupling of the ABJM theory to other background fields could lead to other superspace deformations of the super algebra. So, it will also be interesting to analyse the consequences of coupling of ABJM theory to other background fields.

Chern-Simons theories also have important applications in condensed matter physics. This is because of their relevance to the fractional quantum Hall effect, which is based on the concept of statistical transmutation. [64]-[67]. In two dimensions, fermions can be described as charged bosons carrying an odd integer number of flux quanta. This is achieved by analysing Chern-Simons fields coupled to these bosons. Then, the electrons in an external magnetic field can be described as bosons in a combined external and statistical magnetic field. At special values of the filling fraction the statistical field cancels the external field, in the mean field sense. At these values of the filling fraction the system is described as a gas of bosons feeling no net magnetic field. Thus, these bosons condense into a homogeneous ground state. This model also describes the existence of vortex and anti-vortex excitations.

Lately, supersymmetric generalisation of the fractional quantum Hall effect has also been investigated [68]-[71]. In particular, physical properties of the topological excitations in the supersymmetric quantum Hall liquid have been discussed using a dual supersymmetric Chern-Simons theory [72]. Furthermore, the fractional quantum Hall effect is closely related to noncommutativity of the spacetime [73]-[76]. Thus, the results of this paper can have interesting condensed matter applications. This is because we can analyse the superspace deformation of the supersymmetric fractional quantum Hall effect. It can change the behavior of fractional condensates and thus

have important consequences for the transport properties in the quantum hall system. Holography has also been used to analyse the supersymmetric fractional quantum Hall effect [77]. In fact, supersymmetric Chern-Simons theories have been used to study various interesting examples of AdS_4/CFT_3 correspondence [78]-[82]. It will be interesting to analyse similar effects in the deformed superspace theories with superspace noncommutativity.

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